

ON  $\sigma$ -CONVEX SUBSETS IN SPACES OF SCATTEREDLY CONTINUOUS FUNCTIONS

TARAS BANAKH, BOGDAN BOKALO, NADIYA KOLOS

**ABSTRACT.** We prove that for any topological space  $X$  of countable tightness, each  $\sigma$ -convex subspace  $\mathcal{F}$  of the space  $SC_p(X)$  of scatteredly continuous real-valued functions on  $X$  has network weight  $nw(\mathcal{F}) \leq nw(X)$ . This implies that for a metrizable separable space  $X$ , each compact convex subset in the function space  $SC_p(X)$  is metrizable. Another corollary says that two Tychonoff spaces  $X, Y$  with countable tightness and topologically isomorphic linear topological spaces  $SC_p(X)$  and  $SC_p(Y)$  have the same network weight  $nw(X) = nw(Y)$ . Also we prove that each zero-dimensional separable Rosenthal compact space is homeomorphic to a compact subset of the function space  $SC_p(\omega^\omega)$  over the space  $\omega^\omega$  of irrationals.

This paper was motivated by the problem of studying the linear-topological structure of the space  $SC_p(X)$  of scatteredly continuous real-valued functions on a topological space  $X$ , addressed in [5, 6].

A function  $f : X \rightarrow Y$  between two topological spaces is called *scatteredly continuous* if for each non-empty subspace  $A \subset X$  the restriction  $f|_A : A \rightarrow Y$  has a point of continuity. Scatteredly continuous functions were introduced in [3] (as almost continuous functions) and studied in details in [8], [4] and [7]. If a topological space  $Y$  is regular, then the scattered continuity of a function  $f : X \rightarrow Y$  is equivalent to the weak discontinuity of  $f$ ; see [3], [4, 4.4]. We recall that a function  $f : X \rightarrow Y$  is *weakly discontinuous* if each subspace  $A \subset X$  contains an open dense subspace  $U \subset A$  such that the restriction  $f|_U : U \rightarrow Y$  is continuous.

For a topological space  $X$  by  $SC_p(X) \subset \mathbb{R}^X$  we denote the linear space of all scatteredly continuous (equivalently, weakly discontinuous) functions on  $X$ , endowed with the topology of pointwise convergence. It is clear that the space  $SC_p(X)$  contains the linear subspace  $C_p(X)$  of all continuous real-valued functions on  $X$ . Topological properties of the function spaces  $C_p(X)$  were intensively studied by topologists, see [2]. In particular, they studied the interplay between topological invariants of topological space  $X$  and its function space  $C_p(X)$ .

Let us recall [10, 12] that for a topological space  $X$  its

- *weight*  $w(X)$  is the smallest cardinality of a base of the topology of  $X$ ;
- *network weight*  $nw(X)$  is the smallest cardinality of a network of the topology of  $X$ ;
- *tightness*  $t(X)$  is the smallest infinite cardinal  $\kappa$  such that for each subset  $A \subset X$  and a point  $a \in \bar{A}$  in its closure there is a subset  $B \subset A$  of cardinality  $|B| \leq \kappa$  such that  $a \in \bar{B}$ ;
- *Lindelöf number*  $l(X)$  is the smallest infinite cardinal  $\kappa$  such that each open cover of  $X$  has a subcover of cardinality  $\leq \kappa$ ;
- *hereditary Lindelöf number*  $hl(X) = \sup\{l(Z) : Z \subset X\}$ ;
- *density*  $d(X)$  is the smallest cardinality of a dense subset of  $X$ ;
- *the hereditary density*  $hd(X) = \sup\{d(Z) : Z \subset X\}$ ;
- *spread*  $s(X) = \sup\{|D| : D \text{ is a discrete subspace of } X\}$ .

By [2, §I.1], for each Tychonoff space  $X$  the function space  $C_p(X)$  has weight  $w(C_p(X)) = |X|$  and network weight  $nw(SC_p(X)) = nw(X)$ . For the function space  $SC_p(X)$  the situation is a bit different.

**Proposition 1.** *For any  $T_1$ -space  $X$  we have*

$$s(SC_p(X)) = nw(SC_p(X)) = w(SC_p(X)) = |X|.$$

*Proof.* It is clear that  $s(SC_p(X)) \leq nw(SC_p(X)) \leq w(SC_p(X)) \leq w(\mathbb{R}^X) = |X|$ . To see that  $|X| \leq s(SC_p(X))$ , observe that for each point  $a \in X$  the characteristic function

$$\delta_a : X \rightarrow \mathbb{R} = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{otherwise} \end{cases}$$

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of the singleton  $\{a\}$  is scatteredly continuous, and the subspace  $\mathcal{D} = \{\delta_a : a \in X\} \subset SC_p(X)$  has cardinality  $|X|$  and is discrete in  $SC_p(X)$ .  $\square$

The deviation of a subset  $\mathcal{F} \subset SC_p(X)$  from being a subset of  $C_p(X)$  can be measured with help of the cardinal number  $\text{dec}(\mathcal{F})$  called the *decomposition number* of  $\mathcal{F}$ . It is defined as the smallest cardinality  $|\mathcal{C}|$  of a cover  $\mathcal{C}$  of  $X$  such that for each  $C \in \mathcal{C}$  and  $f \in \mathcal{F}$  the restriction  $f|_C$  is continuous. If the function family  $\mathcal{F}$  consists of a single function  $f$ , then the decomposition number  $\text{dec}(\mathcal{F}) = \text{dec}(\{f\})$  coincides with the decomposition number  $\text{dec}(f)$  of the function  $f$ , studied in [14]. It is clear that  $\text{dec}(C_p(X)) = 1$ .

**Proposition 2.** *For a  $T_1$  topological space  $X$  the decomposition number  $\text{dec}(SC_p(X))$  is equal to the decomposition number  $\text{dec}(\mathcal{D})$  of the subset  $\mathcal{D} = \{\delta_a : a \in X\} \subset SC_p(X)$  and is equal to the smallest cardinality  $\text{ddec}(X)$  of a cover of  $X$  by discrete subspaces.*

*Proof.* It is clear that  $\text{dec}(\mathcal{D}) \leq \text{dec}(SC_p(X)) \leq \text{ddec}(X)$ . To prove that  $\text{dec}(\mathcal{D}) \geq \text{ddec}(X)$ , take a cover  $\mathcal{C}$  of  $X$  of cardinality  $|\mathcal{C}| = \text{dec}(\mathcal{D})$  such that for each  $C \in \mathcal{C}$  and each characteristic function  $\delta_a \in \mathcal{D}$  the restriction  $\delta_a|_C$  is continuous. We claim that each space  $C \in \mathcal{C}$  is discrete. Assuming conversely that  $C$  contains a non-isolated point  $c \in C$ , observe that for the characteristic function  $\delta_c$  of the singleton  $\{c\}$  the restriction  $\delta_c|_C$  is not continuous. But this contradicts the choice of the cover  $\mathcal{C}$ . Therefore the cover  $\mathcal{C}$  consists of discrete subspaces of  $X$  and  $\text{ddec}(X) \leq |\mathcal{C}| = \text{dec}(\mathcal{D})$ .  $\square$

In contrast to the whole function space  $SC_p(X)$  which has large decomposition number  $\text{dec}(SC_p(X))$ , its  $\sigma$ -convex subsets have decomposition numbers bounded from above by the hereditary Lindelöf number of  $X$ .

Following [1] and [15], we define a subset  $C$  of a linear topological space  $L$  to be  $\sigma$ -convex if for any sequence of points  $(x_n)_{n \in \omega}$  in  $C$  and any sequence of positive real numbers  $(t_n)_{n \in \omega}$  with  $\sum_{n=0}^{\infty} t_n = 1$  the series  $\sum_{n=0}^{\infty} t_n x_n$  converges to some point  $c \in C$ . It is easy to see that each compact convex subset  $K \subset L$  is  $\sigma$ -convex. On the other hand, each  $\sigma$ -convex subset of a linear topological space  $L$  is necessarily convex and bounded in  $L$ .

The main result of this paper is the following:

**Theorem 1.** *For any topological space  $X$  of countable tightness, each  $\sigma$ -convex subset  $\mathcal{F} \subset SC_p(X)$  has decomposition number  $\text{dec}(\mathcal{F}) \leq hl(X)$ .*

This theorem will be proved in Section 3. Now we derive some simple corollaries of this theorem.

**Corollary 1.** *For any topological space  $X$  of countable tightness, each  $\sigma$ -convex subset  $\mathcal{F} \subset SC_p(X)$  has network weight  $nw(\mathcal{F}) \leq nw(X)$ . Moreover,*

$$nw(X) = \max\{nw(\mathcal{F}) : \mathcal{F} \text{ is a } \sigma\text{-convex subset of } SC_p(X)\}$$

*provided the space  $X$  is Tychonoff.*

*Proof.* By Theorem 1, each  $\sigma$ -convex subset  $\mathcal{F} \subset SC_p(X)$  has decomposition number  $\text{dec}(\mathcal{F}) \leq hl(X)$ . Consequently, we can find a disjoint cover  $\mathcal{C}$  of  $X$  of cardinality  $|\mathcal{C}| = \text{dec}(\mathcal{F}) \leq hl(X)$  such that for each  $C \in \mathcal{C}$  and  $f \in \mathcal{F}$  the restriction  $f|_C$  is continuous.

Let  $Z = \oplus \mathcal{C} = \{(x, C) \in X \times \mathcal{C} : x \in C\} \subset X \times \mathcal{C}$  be the topological sum of the family  $\mathcal{C}$ , and  $\pi : Z \rightarrow X$ ,  $\pi : (x, C) \mapsto x$ , be the natural projection of  $Z$  onto  $X$ . Since the cover  $\mathcal{C}$  is disjoint, the map  $\pi : Z \rightarrow X$  is bijective and hence induces a topological isomorphism  $\pi^* : \mathbb{R}^X \rightarrow \mathbb{R}^Z$ ,  $\pi^* : f \mapsto f \circ \pi$ . The choice of the cover  $\mathcal{C}$  guarantees that  $\pi^*(\mathcal{F}) \subset C_p(Z)$ . By (the proof of) Theorem I.1.3 of [2],  $nw(C_p(Z)) \leq nw(Z)$  and hence

$$\begin{aligned} nw(\mathcal{F}) &= nw(\pi^*(\mathcal{F})) \leq nw(C_p(Z)) \leq nw(Z) \leq \\ &\leq |\mathcal{C}| \cdot nw(X) \leq hl(X) \cdot nw(X) = nw(X). \end{aligned}$$

If the space  $X$  is Tychonoff, then the “closed unit ball”

$$\mathcal{B} = \{f \in C_p(X) : \sup_{x \in X} |f(x)| \leq 1\} \subset C_p(X)$$

is  $\sigma$ -convex and has network weight  $nw(\mathcal{B}) = nw(X)$  according to Theorem I.1.3 of [2]. So,

$$nw(X) = \max\{nw(\mathcal{F}) : \mathcal{F} \text{ is a } \sigma\text{-convex subset of } SC_p(X)\}.$$

$\square$

In the same way we can derive some bounds on the weight of compact convex subsets in function spaces  $SC_p(X)$ .

**Corollary 2.** *For any topological space  $X$  of countable tightness, each compact convex subset  $\mathcal{K} \subset SC_p(X)$  has weight  $w(\mathcal{K}) \leq \max\{hl(X), hd(X)\}$ . Moreover,*

$$\begin{aligned} hl(X) &\leq \sup\{w(\mathcal{K}) : \mathcal{K} \text{ is a compact convex subset of } SC_p(X)\} \leq \\ &\leq \max\{hl(X), hd(X)\}. \end{aligned}$$

*Proof.* Given a compact convex subset  $\mathcal{K} \subset SC_p(X)$ , use Theorem 1 to find a disjoint cover  $\mathcal{C}$  of  $X$  of cardinality  $|\mathcal{C}| = \text{dec}(\mathcal{K}) \leq hl(X)$  such that for each  $C \in \mathcal{C}$  and  $f \in \mathcal{K}$  the restriction  $f|_C$  is continuous. Let  $Z = \bigoplus \mathcal{C}$  and  $\pi : \bigoplus \mathcal{C} \rightarrow X$  be the natural projection, which induces a linear topological isomorphism  $\pi^* : \mathbb{R}^X \rightarrow \mathbb{R}^Z$ ,  $\pi^* : f \mapsto f \circ \pi$ , with  $\pi^*(\mathcal{K}) \subset C_p(Z)$ . It follows that the topological sum  $Z = \bigoplus \mathcal{C}$  has density  $d(Z) \leq \sum_{C \in \mathcal{C}} d(C) \leq |\mathcal{C}| \cdot hd(X) \leq \max\{hl(X), hd(X)\}$ , and so we can fix a dense subset  $D \subset Z$  of cardinality  $|D| = d(Z) \leq \max\{hl(X), hd(X)\}$ . Since the restriction operator  $R : C_p(Z) \rightarrow C_p(D)$ ,  $R : f \mapsto f|_D$ , is injective and continuous, we conclude that

$$\begin{aligned} w(\mathcal{K}) &= w(\pi^*(\mathcal{K})) = w(R \circ \pi^*(\mathcal{K})) \leq w(\mathbb{R}^D) = \\ &= |D| \cdot \aleph_0 \leq \max\{hl(X), hd(X)\}. \end{aligned}$$

Next, we show that  $hl(X) \leq \tau$  where

$$\tau = \sup\{w(\mathcal{K}) : \mathcal{K} \text{ is a compact convex subset of } SC_p(X)\}.$$

Assuming conversely that  $hl(X) > \tau$  and using the equality  $hl(X) = \sup\{|Z| : Z \subset X \text{ is scattered}\}$  established in [12], we can find a scattered subspace  $Z \subset X$  of cardinality  $|Z| > \tau$ . It is easy to check that each function  $f : X \rightarrow [0, 1]$  with  $f(X \setminus Z) \subset \{0\}$  is scatteredly continuous, which implies that the subset

$$\mathcal{K}_Z = \{f \in SC_p(X) : f(Z) \subset [0, 1], f(X \setminus Z) \subset \{0\}\}$$

is compact, convex and homeomorphic to the Tychonoff cube  $[0, 1]^Z$ . Then  $\tau \geq w(\mathcal{K}_Z) = w([0, 1]^Z) = |Z| > \tau$  and this is a desired contradiction that completes the proof.  $\square$

Corollaries 1 or 2 imply:

**Corollary 3.** *For a metrizable separable space  $X$ , each compact convex subspace  $\mathcal{K} \subset SC_p(X)$  is metrizable.*

Finally, let us observe that Corollary 1 implies:

**Corollary 4.** *If for Tychonoff spaces  $X, Y$  with countable tightness the linear topological spaces  $SC_p(X)$  and  $SC_p(Y)$  are topologically isomorphic, then  $nw(X) = nw(Y)$ .*

## 1. WEAKLY DISCONTINUOUS FAMILIES OF FUNCTIONS

In this section we shall generalize the notions of scattered continuity and weak discontinuity to function families.

A family of functions  $\mathcal{F} \subset Y^X$  from a topological space  $X$  to a topological space  $Y$  is called

- *scatteredly continuous* if each non-empty subset  $A \subset X$  contains a point  $a \in A$  at which each function  $f|_A : A \rightarrow Y$ ,  $f \in \mathcal{F}$  is continuous;
- *weakly discontinuous* if each subset  $A \subset X$  contains an open dense subspace  $U \subset A$  such that each function  $f|_U : U \rightarrow Y$ ,  $f \in \mathcal{F}$  is continuous.

The following simple characterization can be derived from the corresponding definitions and Theorem 4.4 of [4] (saying that each scatteredly continuous function with values in a regular topological space is weakly discontinuous).

**Proposition 3.** *A function family  $\mathcal{F} \subset Y^X$  is scatteredly continuous (resp. weakly discontinuous) if and only if so is the function  $\Delta\mathcal{F} : X \rightarrow Y^{\mathcal{F}}$ ,  $\Delta\mathcal{F} : x \mapsto (f(x))_{f \in \mathcal{F}}$ . Consequently, for a regular topological space  $Y$ , a function family  $\mathcal{F} \subset Y^X$  is scatteredly continuous if and only if it is weakly discontinuous.*

Propositions 4.7 and 4.8 [4] imply that each weakly discontinuous function  $f : X \rightarrow Y$  has decomposition number  $\text{dec}(f) \leq hl(X)$ . This fact combined with Proposition 3 yields:

**Corollary 5.** *For any topological spaces  $X, Y$ , each weakly discontinuous function family  $\mathcal{F} \subset Y^X$  has decomposition number  $\text{dec}(\mathcal{F}) \leq \text{hl}(X)$ .*

## 2. WEAK DISCONTINUITY OF $\sigma$ -CONVEX SETS IN FUNCTION SPACES

For a topological space  $X$  by  $SC_p^*(X)$  we denote the space of all *bounded* scatteredly continuous real-valued functions on  $X$ . It is a subspace of the function space  $SC_p(X) \subset \mathbb{R}^X$ . Each function  $f \in SC_p^*(X)$  has finite norm  $\|f\| = \sup_{x \in X} |f(x)|$ .

**Theorem 2.** *For any topological space  $X$  with countable tightness, each  $\sigma$ -convex subset  $\mathcal{F} \subset SC_p^*(X)$  is weakly discontinuous.*

*Proof.* By Proposition 3, the weak discontinuity of the function family  $\mathcal{F}$  is equivalent to the scattered continuity of the function  $\Delta\mathcal{F} : X \rightarrow \mathbb{R}^{\mathcal{F}}$ ,  $\Delta\mathcal{F} : x \mapsto (f(x))_{f \in \mathcal{F}}$ . Since the space  $X$  has countable tightness, the scattered continuity of  $\Delta\mathcal{F}$  will follow from Proposition 2.3 of [4] as soon as we check that for each countable subset  $Q = \{x_n\}_{n=1}^\infty \subset X$  the restriction  $\Delta\mathcal{F}|_Q : Q \rightarrow \mathbb{R}^{\mathcal{F}}$  has a continuity point. Assuming the converse, for each point  $x_n \in Q$  we can choose a function  $f_n \in \mathcal{F}$  such that the restriction  $f_n|_Q$  is discontinuous at  $x_n$ .

Observe that a function  $f : Q \rightarrow \mathbb{R}$  is discontinuous at a point  $q \in Q$  if and only if it has strictly positive oscillation

$$\text{osc}_q(f) = \inf_{O_q} \sup\{|f(x) - f(y)| : x, y \in O_q\}$$

at the point  $q$ . In this definition the infimum is taken over all neighborhoods  $O_q$  of  $q$  in  $Q$ .

We shall inductively construct a sequence  $(t_n)_{n=1}^\infty$  of positive real numbers such that for every  $n \in \mathbb{N}$  the following conditions are satisfied:

- 1)  $t_1 \leq \frac{1}{2}$ ,  $t_{n+1} \leq \frac{1}{2}t_n$ , and  $t_{n+1} \cdot \|f_{n+1}\| \leq \frac{1}{2}t_n \cdot \|f_n\|$ ,
- 2) the function  $s_n = \sum_{k=1}^n t_k f_k$  restricted to  $Q$  is discontinuous at  $x_n$ ,
- 3)  $t_{n+1} \cdot \|f_{n+1}\| \leq \frac{1}{8} \text{osc}_{x_n}(s_n|_Q)$ .

We start the inductive construction letting  $t_1 = 1/2$ . Then the function  $s_1|_Q = t_1 \cdot f_1|_Q$  is discontinuous at  $x_1$  by the choice of the function  $f_1$ . Now assume that for some  $n \in \mathbb{N}$  positive numbers  $t_1 \dots, t_n$  has been chosen so that the function  $s_n = \sum_{k=1}^n t_k f_k$  restricted to  $Q$  is discontinuous at  $x_n$ .

Choose any positive number  $\tilde{t}_{n+1}$  such that

$$\tilde{t}_{n+1} \leq \frac{1}{2}t_n, \quad \tilde{t}_{n+1} \cdot \|f_{n+1}\| \leq \frac{1}{2}t_n \cdot \|f_n\| \quad \text{and} \quad \tilde{t}_{n+1} \cdot \|f_{n+1}\| \leq \frac{1}{8} \text{osc}_{x_n}(s_n|_Q),$$

and consider the function  $\tilde{s}_{n+1} = s_n + \tilde{t}_{n+1}f_{n+1}$ . If the restriction of this function to  $Q$  is discontinuous at the point  $x_{n+1}$ , then put  $t_{n+1} = \tilde{t}_{n+1}$  and finish the inductive step. If  $\tilde{s}_{n+1}|_Q$  is continuous at  $x_{n+1}$ , then put  $t_{n+1} = \frac{1}{2}\tilde{t}_{n+1}$  and observe that the restriction of the function

$$s_{n+1} = \sum_{k=1}^{n+1} t_k f_k = s_n + \frac{1}{2}\tilde{t}_{n+1}f_{n+1} = \tilde{s}_{n+1} - \frac{1}{2}\tilde{t}_{n+1}f_{n+1}$$

to  $Q$  is discontinuous at  $x_{n+1}$ . This completes the inductive construction.

The condition (1) guarantees that  $\sum_{n=1}^\infty t_n \leq 1$  and hence the number  $t_0 = 1 - \sum_{n=1}^\infty t_n$  is non-negative. Now take any function  $f_0 \in \mathcal{F}$  and consider the function

$$s = \sum_{n=0}^\infty t_n f_n$$

which is well-defined and belongs to  $\mathcal{F}$  by the  $\sigma$ -convexity of  $\mathcal{F}$ .

The functions  $f_0, s \in \mathcal{F} \subset SC_p(X)$  are weakly discontinuous and hence for some open dense subset  $U \subset Q$  the restrictions  $s|_U$  and  $f_0|_U$  are continuous. Pick any point  $x_n \in U$ . Observe that

$$s = t_0 f_0 + s_n + \sum_{k=n+1}^{\infty} t_k f_k$$

and hence

$$s_n = s - t_0 f_0 - \sum_{k=n+1}^{\infty} t_k f_k = s - t_0 f_0 - u_n,$$

where  $u_n = \sum_{k=n+1}^{\infty} t_k f_k$ . The conditions (1) and (3) of the inductive construction guarantee that the function  $u_n$  has norm

$$\|u_n\| \leq \sum_{k=n+1}^{\infty} t_k \|f_k\| \leq 2t_{n+1} \|f_{n+1}\| \leq \frac{1}{4} \text{osc}_{x_n}(s_n|Q).$$

Since  $s_n = s - t_0 f_0 - u_n$ , the triangle inequality implies that

$$\begin{aligned} 0 < \text{osc}_{x_n}(s_n|Q) &\leq \text{osc}_{x_n}(s|Q) + \text{osc}_{x_n}(t_0 f_0|Q) + \text{osc}_{x_n}(u_n) \leq \\ &\leq 0 + 0 + 2\|u_n\| \leq \frac{1}{2} \text{osc}_{x_n}(s_n|Q) \end{aligned}$$

which is a desired contradiction, which shows that the restriction  $\Delta\mathcal{F}|Q$  has a point of continuity and the family  $\mathcal{F}$  is weakly discontinuous.  $\square$

### 3. PROOF OF THEOREM 1

Let  $X$  be a topological space with countable tightness and  $\mathcal{F}$  be a  $\sigma$ -convex subset in the function space  $SC_p(X)$ . The  $\sigma$ -convexity of  $\mathcal{F}$  implies that for each point  $x \in X$  the subset  $\{f(x) : f \in \mathcal{F}\} \subset \mathbb{R}$  is bounded (in the opposite case we could find sequences  $(f_n)_{n \in \omega} \in \mathcal{F}^\omega$  and  $(t_n)_{n \in \omega} \in [0, 1]^\omega$  with  $\sum_{n=0}^{\infty} t_n = 1$  such that the series  $\sum_{n=1}^{\infty} t_n f_n(x)$  is divergent). Then  $X = \bigcup_{n=1}^{\infty} X_n$  where  $X_n = \{x \in X : n \leq \sup_{f \in \mathcal{F}} |f(x)| < n+1\}$  for  $n \in \omega$ .

It follows that for every  $n \in \omega$  the family  $\mathcal{F}|X_n = \{f|X_n : f \in \mathcal{F}\}$  is a  $\sigma$ -convex subset of the function space  $SC_p^*(X_n)$ . By Theorem 2, the function family  $\mathcal{F}|X_n$  is weakly discontinuous and by Corollary 5,  $\text{dec}(\mathcal{F}|X_n) \leq hl(X_n)$ . Then  $\text{dec}(\mathcal{F}) \leq \sum_{n=0}^{\infty} \text{dec}(\mathcal{F}|X_n) \leq \sum_{n=0}^{\infty} hl(X_n) \leq hl(X)$ .

### 4. SOME OPEN PROBLEMS

The presence of the condition of countable tightness in Theorem 1 and its corollaries suggests the following open problem.

**Problem 1.** *Is it true  $w(\mathcal{K}) \leq nw(X)$  for each topological space  $X$  and each compact convex subset  $\mathcal{K} \subset SC_p(X)$ ?*

By Theorem 2, for each topological space  $X$  of countable tightness, each compact convex subset  $\mathcal{K} \subset SC_p^*(X)$  is weakly discontinuous.

**Problem 2.** *For which topological spaces  $X$  each compact convex subset  $\mathcal{K} \subset SC_p(X)$  is weakly discontinuous?*

According to Corollary 3, each compact convex subset  $\mathcal{K} \subset SC_p(\omega^\omega)$  is metrizable.

**Problem 3.** *Is a compact subset  $\mathcal{K} \subset SC_p(\omega^\omega)$  metrizable if  $\mathcal{K}$  is homeomorphic to a compact convex subset of  $\mathbb{R}^\mathbb{C}$ .*

Let us recall that a topological space  $K$  is *Rosenthal compact* if  $K$  is homeomorphic to a compact subspace of the space  $\mathcal{B}_1(X) \subset \mathbb{R}^X$  of functions of the first Baire class on a Polish space  $X$ . In this definition the space  $X$  can be assumed to be equal to the space  $\omega^\omega$  of irrationals.

**Problem 4.** *Is each Rosenthal compact space homeomorphic to a compact subset of the function space  $SC_p(\omega^\omega)$ ?*

This problem has affirmative solution in the realm of zero-dimensional separable Rosenthal compacta.

**Theorem 3.** *Each zero-dimensional separable Rosenthal compact space  $K$  is homeomorphic to a compact subset of the function space  $SC_p(\omega^\omega)$ .*

*Proof.* Let  $D \subset K$  be a countable dense subset in  $K$ . Let  $A = C_D(K, 2)$  be the space of continuous functions  $f : K \rightarrow 2 = \{0, 1\}$  endowed with the smallest topology making the restriction operator  $R : C_D(K, 2) \rightarrow 2^D$ ,  $R : f \mapsto f|_D$ , continuous. By the characterization of separable Rosenthal compacta [11], the space  $A$  is analytic, i.e.,  $A$  is the image of the Polish space  $X = \omega^\omega$  under a continuous map  $\pi : X \rightarrow A$ . Now consider the map  $\delta : K \rightarrow 2^A$ ,  $\delta : x \mapsto (f(x))_{f \in A}$ . This map is continuous and injective by the zero-dimensionality of  $K$ . The map  $\pi : X \rightarrow A$  induces a homeomorphism  $\pi^* : 2^A \rightarrow 2^X$ ,  $\pi^* : f \mapsto f \circ \pi$ . Then  $\pi^* \circ \delta : K \rightarrow 2^X$  is a topological embedding.

We claim that  $\pi^* \circ \delta(K) \subset SC_p(X) \cap 2^X$ . Given a point  $x \in K$ , we need to check that the function  $\pi^* \circ \delta(x) \in 2^X$  is scatteredly continuous. It will be convenient to denote the function  $\delta(x) \in 2^A$  by  $\delta_x$ . This function assigns to each  $f \in A = C_D(K)$  the number  $\delta_x(f) = f(x) \in 2$ .

By [13, 9], the Rosenthal compact space  $K$  is Fréchet-Urysohn, so there is a sequence  $(x_n)_{n \in \omega} \in D^\omega$  with  $\lim_{n \rightarrow \infty} x_n = x$ . Then the function  $\delta_x : A \rightarrow 2$ ,  $\delta_x : f \mapsto f(x)$ , is the pointwise limit of the continuous functions  $\delta_{x_n}$ , which implies that  $\delta_x$  is a function of the first Baire class on  $A$  and  $\delta_x \circ \pi : X \rightarrow 2$  is a function of the first Baire class on the Polish space  $X$ . Since this function has discrete range, it is scatteredly continuous by Theorem 8.1 of [4]. Consequently,  $\pi^* \circ \delta(x) \in SC_p(X)$  and  $K$  is homeomorphic to the compact subset  $\pi^* \circ \delta(K) \subset SC_p(X)$ .  $\square$

A particularly interesting instance of Problem 4 concerns non-metrizable convex Rosenthal compacta. One of the simplest spaces of this sort is the Helly space. We recall that the *Helly space* is the subspace of  $B_1(I)$  consisting of all non-decreasing functions  $f : I \rightarrow I$  of the unit interval  $I = [0, 1]$ .

**Problem 5.** *Is the Helly space homeomorphic to a compact subset of the function space  $SC_p(\omega^\omega)$ ?*

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IVAN FRANKO NATIONAL UNIVERSITY OF LVIV

E-mail address: tbanakh@yahoo.com, bogdanbokalo@mail.ru, nadiya\_kolos@ukr.net